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Approximating voting rules from truncated ballots

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Abstract

Classical voting rules assume that ballots are complete preference orders over candidates. However, when the number of candidates is large enough, it is too costly to ask the voters to rank all candidates. We suggest to fix a rank k, to ask all voters to specify their best k candidates, and then to consider "top-k approximations" of rules, which take only into account the *top-k* candidates of each ballot. The questions are then: *Are these k-truncated approximations good predictors of the approximated rule? For which values of k and under which assumptions can we expect to output the correct winner with high probability?* For different voting rules, we study these questions theoretically, by giving tight approximation ratios, and empirically, based on randomly generated profiles and on real data. We consider two measures of the quality of the approximation: the probability of selecting the same winner as the original rule, and the score ratio. We do a worst-case study (for the latter measure only), and for both measures, an average-case study and a study from real data sets.

Keywords Voting rules · Truncated ballots · Approximations

1 Introduction

Voting consists of aggregating voters' preferences over a set of candidates in order to determine a consensus decision or recommendation.

A voting rule maps a *profile* into a collectively chosen candidate. The precise definition of a profile varies across voting rules. While some rules start from approval profiles (each vote being a subset of candidates) and other start from quantitative profiles (each vote associating a numerical score with each candidate), most rules that have been studied in the

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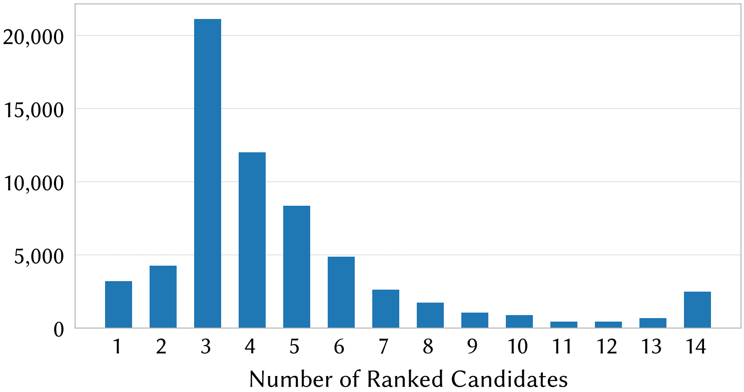


Fig. 1 In the 2002 Irish election for meath constituency, most of the voters rank only 3 to 5 candidates out of 14

literature start from *ordinal profiles*, where each vote is a linear order, i.e., a ranking, over candidates. In this paper we focus on the latter family of rules.

Ranking candidates has many advantages: it allows voter to express much more information than with an approval ballot, and it avoids the well-known issue of interpersonal comparison of preferences that makes quantitative profiles difficult to work with. However, it comes with one difficulty: requiring a voter to provide a complete ranking over the whole set of candidates can be difficult and costly in terms of time and cognitive effort. Indeed voters can find it difficult and/or time-consuming to rank all candidates running up for election when the number of candidates is more than a very small number (such as 4 or 5). A way to palliate this problem, already argued in a number of works, consists in asking voters to report *truncated ballots*: each voters reports only their *top*-k candidates, for some (small) fixed value of k.

In practice, voting systems often permit voters to report truncated ballots. For example, in the 2002 Irish Election for Meath constituency (full ballot data are available on [25]), most voters chose to rank between 3 and 5 of the 14 candidates, with only 3.89% of voters submitting a full ranking (see Fig. 1).

However, this raises the issue of how common voting rules should be adapted to top-k ballots. We generalize the definition of a voting rule such that it takes truncated ballots as input. We instantiate this definition on several common voting rules. For each of them the variant with truncated ballots as input can be seen either as a genuinely novel rule, or as an approximation of the original rule. We mostly take the second view and then ask, *are these approximations good predictors of the original rule?*

We answer this question by considering two measures of the quality of the approximation: (1) the probability of selecting the same winner as the original rule, and (2) the worst-case score ratio. Which one of these two measures is more relevant depends on the application domain. In epistemic social choice (see for instance part 2 of [11]), the final aim is to uncover the ground truth, therefore measure (1) is more relevant: if the voting rule is a maximum likelihood estimator for the chosen voting rule, it makes sense to maximize the probability that the truncated approximation will find the same outcome of the voting rule. In classical social choice where voters have hidden cardinal preferences, and where the score function defining the rule is considered a proxy for their utilities, then the global score of an alternative can be considered a proxy for the social welfare of this alternative; the worst-case ratio between the score of the outcome of the initial voting rule and the score of the outcome of the approximation is then a classical "price of anarchy" type of measure, which would be called here *price of truncation*. We do a theoretical analysis only for measure (1). For both measures (1) and (2) we make an empirical study, based on randomly generated profiles and on real-world data. Our findings are that for several common voting rules, both for randomly generated profiles and real data, a very small k suffices.

Our interpretation of top-k ballots is *epistemic*: the central authority in charge of collecting the votes and computing the outcome ignores the voters' preferences below the *top*-k candidates of each voter, and has to cope with it as much as possible. Voters may very well have a complete preference order in their head (although it does not need to be the case), but they will simply not be asked to report it.

Section 3 gives some background on social choice and voting rules. Section 4 defines top-k approximations of different voting rules. Section 5 analyses empirically the probability that approximate rules select the true winner. Finally, Sect. 6 analyses score distortion, theoretically and empirically.

2 Related work

Voting with truncated ballots is a form of voting with incomplete knowledge (a background on this more general topic can be found in [5]). Existing work on truncated ballots can be classified into two classes according to the type of interaction with the voters.

2.1 Interactive elicitation with top-*k* ballots

An interactive elicitation protocol asks voters to expand their truncated ballots in an incremental way, until the outcome of the vote is eventually determined. This line of research starts with Kalech et al. [18] who start by top-1 ballots, then top-2, etc., until there is sufficient information for knowing the winner. Lu and Boutilier [22, 23] propose an incremental elicitation process using *minimax regret* to predict the correct winner given partial information. A more general incremental elicitation framework, with more types of elicitation questions, is cost-effective elicitation [32]. Naamani Dery et al. [10] present two elicitation algorithms for finding a winner with little communication between voters.

2.2 Non-interactive elicitation with top-*k* ballots

Here the central authority elicits the top-k ballots at once, for a fixed value of k, and outputs a winner without requiring voters to provide extra information.

A possibility consists in computing possible winners given these truncated ballots: this is the path followed by Baumeister et al. [2], who also consider double-truncated ballots where voters rank some of their top and bottom candidates. Even if outputting possible winners can also be seen as a way of generalizing the definition of a voting rule to truncated ballots, the obtained rule tends to be very irresolute when the size of the partial ballots is small.

Another possibility – which is the one we follow – consists in generalizing the definition of a voting rule so that it takes truncated ballots as input. A few works go along this line; we discuss them now.

Bentert and Skowron [3] focus on the top-k approximations of two voting rules: Borda and maximin. They measure the extent to which these top-k rules approximate the original Borda (resp. maximin) rule by the worst-case ratio between the Borda (resp. maximin) scores, with respect to the original profile, of the winner of the original rule and the winner of the top-k variant. Also, they identify the top-k rules that best approximate positional scoring rules. Their theoretical analysis is complemented by numerical experiments using profiles generated from different distributions over preferences. More details will be given in Sect.6.

Oren et al. [27] analyze *top-k* voting by assessing the values of k needed to ensure the true winner is found with high probability for specific preference distributions. Filmus and Oren [13] study the performance of top-k voting under the impartial culture distribution for the Borda, Harmonic and Copeland rules. They assess the values of k needed to find the true winner with high probability, and more precisely, they show that for Borda and the Harmonic (resp. Copeland) voting, a lower bound of $k = \Omega(m)$ (resp. $k = \Omega\left(\frac{m}{\sqrt{\log m}}\right)$) is needed for n sufficiently large relative to m (where n is the number of voters and m is the number of candidates). Their theoretical analysis is complemented by numerical experiments that show that under the impartial culture, in the setting where $m = 20$ and $n = 2000$, Harmonic rule gives the best results where $k = 15$ out of 20 is sufficient to determine the winner, while for Copeland and Borda the whole profile is needed to ensure the winner. Our Sect. 5 can be seen as a continuation of [13]. We go further on several points: we consider more voting rules; beyond impartial culture, we consider a large scope of distributions; we study score distortion; and we include experiments using real-world data sets.

Ayadi et al. [1] evaluate the extent to which STV with *top-k* ballots approximates STV with full information. They show that for small k, *top-k* ballots are enough to identify the correct winner quite frequently, especially for data taken from real elections. Finally, the recognition of singled-peaked *top-k* profiles is studied in [20] while the computational issues of manipulating rules with *top-k* profiles is addressed in [26].

Terzopoulou and Endriss [30] give a thorough axiomatic study of different versions of the Borda rule for truncated ballots. Beyond single-winner rules, top-k approximations also make sense for multi-winner rules. Skowron *et al.* [29] use *top-k* voting as a way to approximate some multiwinner rules.

3 Preliminaries

An election is a triple $E = \langle N, A, P\rangle$ where: $N = \{1, ..., n\}$ is the set of *voters*, A is the set of *candidates*, with $|A| = m$; and $P = (\succ_1, ..., \succ_n)$ is the *preference profile* of voters in N, where for each i, $\succ_i \in P$ is a linear order over A. $\mathcal{P}_m$ is the set of all profiles over m alternatives (for varying n). For any $a, b \in A$, $a \succ_i b$ means that voter i prefers a to b.

Given a profile P, $N_P(a, b) = \#\left\{i, a \succ_i b\right\}$ is the number of voters who prefer a to b in P. The *majority graph* $M(P)$ is the graph whose set of vertices is A and in which for all $a, b \in A$, there is a directed edge from a to b (denoted by $a \rightarrow b$) in $M(P)$ if $N_p(a, b) > \frac{n}{2}$.

Given a profile P, the *weighted majority graph* associated with P is the graph $MW(P)$ whose set of vertices is A and in which for all $a, b \in A$, there is a directed edge from a to b weighted by the number of voters who prefer a to b in P. Since the votes in P are linear orders, knowing $MW(P)$ is equivalent to knowing the pairwise majority matrix defined by: for all $a, b \in A$, $Score_P(a, b) = N_P(a, b) - N_P(b, a)$

A resolute voting rule is a function $f : E \rightarrow A$. Resolute rules are typically obtained from composing an irresolute rule (mapping an election into an non-empty subset of candidates, called co-winners) with a tie-breaking mechanism. For most of the rules below we define the irresolute version.

Positional scoring rules: A *positional scoring rule* (*PSR*) $f^{\mathbf{s}}$ is defined by a non-negative vector $\mathbf{s} = \left(s_1, ..., s_m\right)$ such that $s_1 \geq ... \geq s_m$ and $s_1 > 0$. Each candidate receives s_j points from each voter i who ranks her in the *jth* position, and the score of a candidate is the total number of points she receives from all voters i.e. $S(x) = \sum_{i=1}^{n} s_j$. The winner is the candidate with highest total score. Examples of scoring rules are the Borda and Harmonic rules, with $\mathbf{s}_{\text{Borda}} = (m-1, m-2, \dots, 0)$ and $\mathbf{s}_{\text{Harmonic}} = (1, \frac{1}{2}, \dots, \frac{1}{m})$.

Single Transferable Vote (STV): Given a prespecified linear order $\rhd$ over the candidates (needed for tie-breaking), the $STV^{\rhd}$ rule proceeds in rounds (up to $m-1$): in each round we compute, for each candidate, the number of voters who rank it first. The candidate with the smallest number of voters ranking them first is eliminated and the votes who supported it now support their preferred candidate among those that remain. If there is a tie between two or more candidates, the eliminated candidate is the one, among those with the smallest number of votes, that has the lowest priority according to $\rhd$. The last remaining alternative is the STV winner.

Pairwise comparison rules: They are defined from the majority graph or from the weighted majority graph induced from the profile. We will study three of them:

- The *Copeland* rule outputs the candidate maximizing the *Copeland score*, where the Copeland score of x is the number of candidates y with $x \rightarrow y$ in $M(P)$, plus half the number of candidates $y \neq x$ with no edge between x and y in $M(P)$. The winner is the candidate with highest Copeland score.
- The *Ranked Pairs* (RP) rule proceeds by ranking all pairs of candidates (x, y) according to $N_P(x, y)$ (using tie-breaking when necessary); starting from an empty graph over A, it then considers all pairs in the described order and includes a pair in the graph if and only if it does not create a cycle in it. At the end of the process, the graph is a complete ranking, whose top element is the winner (see Section 4.5.3 of [14] for a formal definition).
- The *maximin* rule outputs the candidates that maximize

$$S_m(x) = min_{x \in A(y \neq x)}\left(N_P(x, y)\right)$$

For the experiments using randomly generated profiles, we use the *Mallows ϕ-model* [24]. It is a (realistic) family of distributions over rankings, parametrized by a modal or reference ranking σ and a dispersion parameter $\phi \in [0, 1]$: $P(r; \sigma, \phi) = \frac{1}{Z}\phi^{d(r,\sigma)}$, where r is any ranking, d is the Kendall tau distance and $Z = \sum_{r'} \phi^{d\left(r',\sigma\right)} = 1 \cdot (1+\phi) \cdot \left(1+\phi+\phi^2\right) \cdot ... \cdot \left(1+...+\phi^{m-1}\right)$ is a normalization constant. With small values of ϕ, the mass is concentrated around σ, while $\phi = 1$ gives the uniform distribution *Impartial Culture (IC)*, where all profiles are equiprobable.

To overcome the unimodal nature of Mallows ϕ model, mixtures of Mallows will be considered. Let p be a positive integer, a mixture model consists of p Mallows models with a probability distribution over them. Formally, given reference rankings $\sigma_1, \dots, \sigma_p$, dispersion parameters $\phi_1, \dots, \phi_p$, and mixing coefficients (discrete probability distribution) $\lambda_1, \dots, \lambda_p$ where each λ_i, $1 \leq i \leq p$ is between 0 and 1, and $\sum_{i=1}^{p} \lambda_p = 1$; we generate rankings with $d = (\sigma, \sigma')$ from the reference ranking that is proportional to ϕ_i^d. We select rankings from the p models according to the given probability distribution [21, 24].

4 Approximating voting rules from truncated ballots

Given $k \in \{1, ..., m-1\}$, a *top-k* election is a triple $E' = \langle N, A, R\rangle$ where N and A are as before, and $R = (\succ_1^k, ..., \succ_n^k)$, where each $\succ_i^k$ is a ranking of k out of m candidates in A. R is called a *top-k profile*. If P is a complete profile, $\succ_i^k$ is the top-k truncation of $\succ_i$ (i.e., the best k candidates, ranked as in $\succ_i$), and $P_k = (\succ_1^k, ..., \succ_n^k)$ is the top-k-profile induced from P and k. A *top-k (resolute) voting rule* is a function f_k that maps each *top-k* election E' to a candidate in A. We sometimes apply a top-k rule to a complete profile, with $f_k(P) = f_k(P_k)$. We now define several *top-k* rules.

4.1 Positional scoring rules

We first generalize the definition of a positional scoring rule to top-k ballots.

Definition 1 A *top-k* PSR $f^{\mathbf{s}_k}$ is defined by a scoring vector $\mathbf{s} = \left(s_1, \ldots, s_k, s^*\right)$ such that $s_1 \geq s_2 \geq ... \geq s_k \geq s_{k+1} \geq s^* \geq 0$ and $s_1 > s^*$. Each candidate in a *top-k* vote receives s_j points from each voter i who ranks her in the *jth* position. A non-ranked candidate gets s^* points. The winner is the candidate with highest total score.

The reason why we require $s^* \in [0, s_{k+1}]$ is that a candidate that is not in the top-k is in the best case in position $k+1$ (and would thus get s_{k+1} points if its position were known) and in the worst case in position m (and would thus get $s_m = 0$ point if its position were known).

When starting from a specific *PSR* for complete ballots, defined by scoring vector $\mathbf{s} = \left(s_1, \ldots, s_m\right)$, two choices of s^* particularly make sense:

- zero score: $s^* = 0$
- average score: $s^* = \frac{1}{m-k}\left(s_{k+1} + \ldots + s_m\right)$

We denote the corresponding approximate rules as f_k^0 and f_k^{av}. $Borda_k^{av}$ is known under the name *average score modified Borda Count* [8, 16], while $Borda_k^0$ is known under the name *modified Borda Count* [12]).

Example 1 Let $m = 5$ and $k = 2$. The scoring vectors for $Borda_2^0$ and $Borda_2^{av}$ are: $\mathbf{s}_{Borda_2^0} = (4, 3, 0, 0, 0)$ and $\mathbf{s}_{Borda_2^{av}} = (4, 3, 1, 1, 1)$, respectively.

The scoring vectors for $Harmonic_2^0$ and $Harmonic_2^{av}$ are: $\mathbf{s}_{Harmonic_2^0} = (1, \frac{1}{2}, 0, 0, 0)$ and $\mathbf{s}_{Harmonic_2^{av}} = (1, \frac{1}{2}, 0.26, 0.26, 0.26)$, respectively.

Young [31] characterized positional scoring rules by these four properties, which we describe informally (for resolute rules):

- *Neutrality*: all candidates are treated equally.
- *Anonymity*: all voters are treated equally.
- *Reinforcement*: if P and Q are two profiles (on disjoint electorates) and x is the winner for P and the winner for Q, then it is also the winner for $P \cup Q$.
- *Continuity*: if P and Q are two profiles and x is the winner for P but not for Q, adding sufficiently many votes of P to Q leads to elect x.

f is a *PSR* if and only if it satisfies neutrality, anonymity, reinforcement and continuity [31]. These four properties still make sense for truncated ballots.

We now define a property of standard voting rules (with complete profiles as input): a standard voting rule is *top-k-only* if for any two complete profiles P, P', if $P_k = P'_k$, then $f(P) = f(P')$. The special case for $k = 1$ is known under the name *tops-only*.

Lemma 1 *A standard positional scoring rule f associated with score vector is top-k-only if and only if* $s_{k+1} = \ldots = s_m$.

Proof The 'if' direction is obvious.

For the 'only if' direction, suppose the equality is not satisfied: then $s_{k+1} > s_m$. We construct two profiles P, P' such that $P_k = P'_k$ and $f(P) \neq f(P')$. Let the candidates be $\{x_1, \ldots, x_{m-1}, y, z\}$. Let α be some integer, which we will specify later. P contains $(2\alpha + 1).(m-2)$ votes:

- For each $i = 1, \ldots m-2$, α votes $yzx_i \ldots x_{i+1[m-2]}, \ldots x_{i-1[m-2]}$, where $[m-2]$ mean "modulo $m-2$";
- For each $i = 1, \ldots m-2$, α votes $zyx_i \ldots x_{i+1[m-2]}, \ldots x_{i-1[m-2]}$
- For each $i = 1, \ldots m-2$, one vote $x_i \ldots x_{i+k-1[m-2]} y x_{i+k[m-2]} \ldots x_{i-1[m-2]} z$.

When α grows, for each $i = 1, \ldots m-2$, $S(x_i) = 2\alpha(s_3 + \ldots + s_m) + s_1 + \ldots + s_{k-1} + s_{k+1} + \ldots + s_m$ while $S(y) = \alpha(m-2)(s_1 + s_2) + (m-2)s_{k+1}$ and $S(y) = \alpha(m-2)(s_1 + s_2) + (m-2)s_m$. Now, because $s_1 > s_m$, we have $s_1 + s_2 > 2\alpha\frac{s_3 + \ldots + s_m}{m}$, which implies that for α large enough, $S(y)$ and $S(z)$ are both larger than $S(x_i)$ for all i. Finally, $S(y) > S(z)$ because of our assumption $s_{k+1} > s_m$. Therefore, the winner in P is y.

Now, let P' the profile identical to P but exchanging the positions of y and z. We have $P_k = P'_k$ and $f(P') = z \neq f(P) = y$. □

Now it is not difficult to generalize Young's result to *top-k* PSR[1].

Theorem 1 *A top-k voting rule is a top-k PSR if and only if it satisfies neutrality, anonymity, reinforcement, and continuity.*

Proof The left-to-right direction is obvious. For the right-to-left direction: assume f_k is a *top-k* rule satisfying neutrality, anonymity, reinforcement, and continuity. Let f be the standard voting rule defined by $f(P) = f_k(P_k)$. Clearly, f also satisfies neutrality, anonymity, reinforcement, and continuity, and due to Young's characterization result, f is a PSR, associated with some vector $(s_1, \ldots, s_m)$. Because f is also *top-k*-only, using Lemma 1 we have $s_{k+1} = \ldots = s_m$, therefore, f_k is a *top-k*-PSR. □

4.2 Rules based on pairwise comparisons

Given a truncated ballot $\succ_i^k$ and two candidates $a, b \in A$, we say that a dominates b in $\succ_i^k$, denoted by $a >_i^k b$, if one of these two conditions holds:

[1] A more general version of this result was proven by Kruger and Terzopoulou [19].

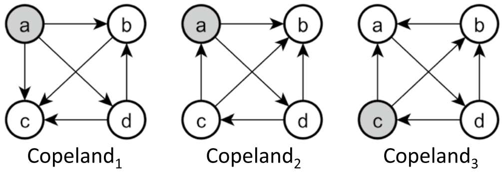


Fig. 2 k-truncated majority graph and k-Copeland for $k = \{1, 2, 3\}$

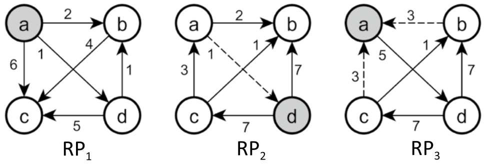


Fig. 3 k-truncated weighted majority graph and k-RP for $k = \{1, 2, 3\}$

1. a and b are listed in $\succ_i^k$ and $a \succ_i^k b$, or
2. a is listed in $\succ_i^k$ and b is not.

For instance, for $A = \{a, b, c, d\}$, $k = 2$, and $\succ_i^2 = (a \succ b)$, then a dominates b (condition 1 is satisfied), both a and b dominate c and d (condition 2 is satisfied), but c and d remain incomparable in $\succ_i^2$.

Now, the notions of pairwise comparison and majority graph are extended to *top-k* truncated profiles in a straightforward way:

Definition 2 k-truncated majority graph Given a top-k profile R, $N_R(a, b) = \#\{i, a >_i^k b\}$ is the number of voters in R for whom a dominates b. The *top-k majority graph* $M_k(R)$ induced by R is the graph whose set of vertices is A and in which there is a directed edge from a to b if $N_R(a, b) > N_R(b, a)$.

Definition 3 k-truncated weighted majority graph Given a k-truncated profile R, for any two candidates a, b, $N_R(a, b) = \#\{i, a >_i^k b\}$ is the number of voters in R for whom a dominates b. The *k-truncated pairwise majority matrix* (or *k-truncated weighted majority graph*) $MW_k(R)$ is the $m \times m$ matrix defined by $MW_k(R)(a, b) = N_R(a, b) - N_R(b, a)$.

Then, given a k-truncated profile, the truncated voting rules $Copeland_k$, $maximin_k$ and RP_k are defined exactly as their standard counterparts *Copeland*, *maximin* and *RP*, starting from the k-truncated (weighted or unweighted) majority graph instead of the standard one. Note that $f_{m-1} = f$, and (for all rules f we consider) f_1 coincides with plurality.

Example 2 Let us consider this profile R with 15 voters and 4 candidates.

$$\begin{array}{l|l} 3\,votes\;\; a \succ d \succ c \succ b & 5\,votes\;\; d \succ c \succ b \succ a \\ 3\,votes\;\; a \succ d \succ b \succ c & 4\,votes\;\; b \succ c \succ a \succ d \end{array}$$

For all rules considered, the tie-breaking priority order is $a \rhd b \rhd c \rhd d$: Figure 2 depicts the k-truncated majority graph for $k = \{1, 2, 3\}$; the Copeland winner is shaded. Figure 3 depicts the k-truncated weighted majority graph: for each pair of candidates (x, y) such that $N_R(x, y) \geq \frac{n}{2}$, the value of the edge is $N_R(x, y)$; the RP$_k$ winner for $k \in \{1, 2, 3\}$ is shaded.

The dashed edges are those that create cycles when running the algorithm that determines the RP_k winner. The $maximin_k$ winners for $k = 1, 2, 3$ coincide with the RP_k-winners.

4.3 STV_k

The STV_k rule is defined by Ayadi et al. [1] as follows: For each $1 \leq k \leq m$, just like STV, in each round, the candidate ranked first by the smallest number of voters is eliminated (breaking ties using $\rhd$ if necessary). If a vote has all its k candidates eliminated, it is said to be *exhausted*, and will be ignored in later rounds. This process is repeated until one candidate remains, who is the winner according to STV_k.

Example 3 With the preference profile of Example 2, the winner of STV_k for all $k \in \{1, 2, 3\}$ is always a. For instance, for STV_2, c is eliminated first then b and the four votes ($b \succ c$) are exhausted. d is eliminated next, the four votes ($d \succ c$) are exhausted and a wins.

5 Evaluation of the probability of selecting the true winner

We will consider two ways of measuring the quality of the *top-k* approximations. The first one (in this Section) is the probability that they output the *true winner*, that is, the winner of the original voting rule, under various distributions and for real-world data. The second one (in the next Section) will be the worst-case ratio between the scores of the winner of the original rule and the winner of the truncated approximation of the rule.

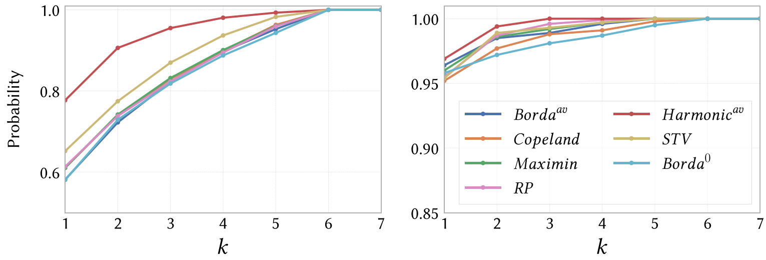
Note that Filmus and Oren [13] go along the first way while Bentert and Skowron [3] go along the second way. Our aim, in this Section, is to see the effect of various parameters of the probability of a top-k approximation f_k of a voting rule f selecting the true winner (which we call the *accuracy* of f_k). We ask the following questions:

- How easy are the various rules to approximate by their top-k version?
- How does the accuracy of f_k evolve with the number of voters n?
- How does the accuracy of f_k evolve with the length of ballots k? Depending on the other parameters, what is the minimal length of the truncated ballots such that the accuracy is close enough to 1?
- How does the accuracy of f_k evolve with the correlation between the voters?
- are these trends similar for all considered voting rules?

These questions may not always be easy to answer independently, since the effects of the various parameters may not always be independent.

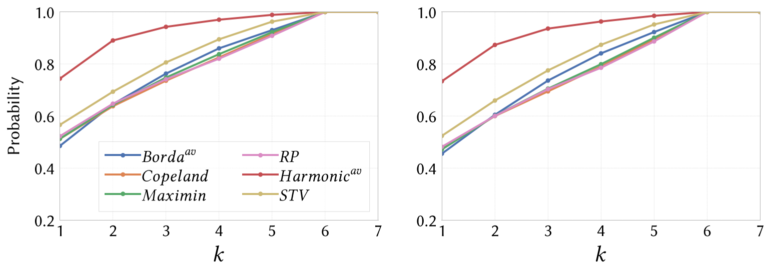
5.1 The experimental set-up

To measure empirically the accuracy of various f_k rules, we repeatedly apply these two steps until obtaining meaningful results:



(a) $\phi = 0.8$ $n = 15$ **(b)** $\phi = 0.8$ $n = 500$

Fig. 4 Mallows, $\phi = 0.8$, $m = 7$, varying k and n



(a) $\phi = 0.9$ **(b)** $\phi = 1$ (IC)

Fig. 5 Mallows, $m = 7$, $n = 15$, varying ϕ and k

1. Generate a complete profile P with n voters and m candidates. The generation method depends on whether we work with randomly generated data or real data. In the former case, we sample a profile from a given distribution. In the latter case, we select n votes uniformly at random from the data set.
2. Compare $f(P)$ to $f_k(P_k)$ for each $k = \{1, \ldots, m-2\}$.

Our experimental setup is similar to that of Filmus and Oren [13]. However, we consider many more rules, and beyond *Impartial Culture* we also consider correlated distributions, using the Mallows model and mixtures of Mallows models. In all experiments with a small number of voters i.e. $n < 1000$ we draw 10,000 random preference profiles while with a large number of voters[2] we ran only 1000 profiles.

All figures show the fraction of profiles on which the top-k rule (for various values of k) selects the correct winner.

Figures 4, 5, 6, 7 and 8 all concern results with the Mallows model. More specifically:

1. In Fig. 4 we vary the number of voters $n \in \{15, 500\}$, with a fixed value of $\phi = 0.8$ and $m = 7$.

[2] Experiments in Fig. 10 (when $n^* > 100$), Figs. 6 and 8.

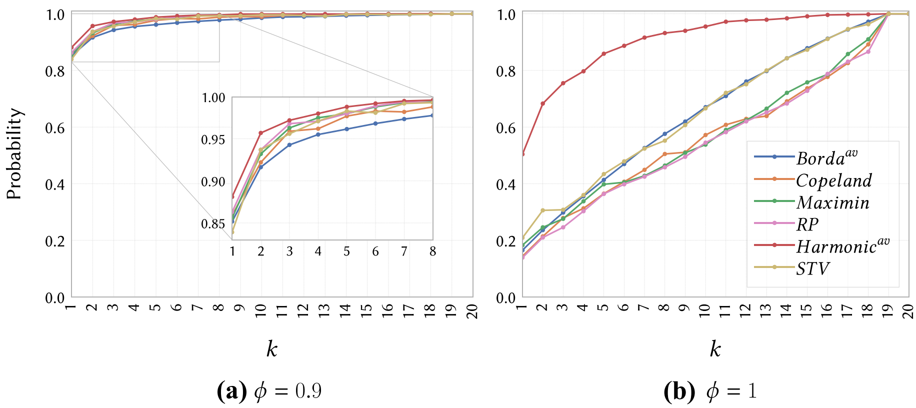


Fig. 6 Mallows, $n = 2000$, $m = 20$, varying ϕ and k

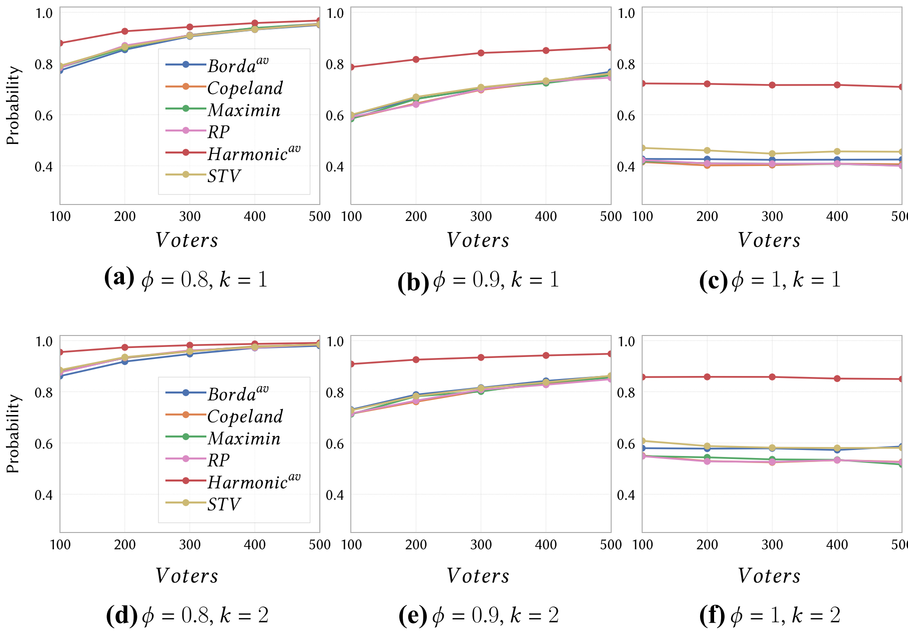


Fig. 7 Mallows, $m = 7$, $k = \{1, 2\}$ varying n and ϕ

2. In Fig. 5 (resp. Fig. 6) we vary the value of dispersion parameter $\phi \in \{0.9, 1\}$, with a fixed value of $m = 7$ (resp. $m = 20$), and a fixed value of $n = 15$ (resp. $n = 2000$) (small for Fig. 5 , large for Fig. 6).
3. In Fig. 7 we vary $n \in \{100, \dots, 500\}$ and $\phi \in \{0.8, 0.9, 1\}$, with a fixed $m = 7$, and focus on $k = 1$ and $k = 2$.
4. In Fig. 8 we vary $m \in \{7, 10, 15, 20\}$ and $\phi \in \{0.9, 1\}$ with a fixed $n = 1000$.

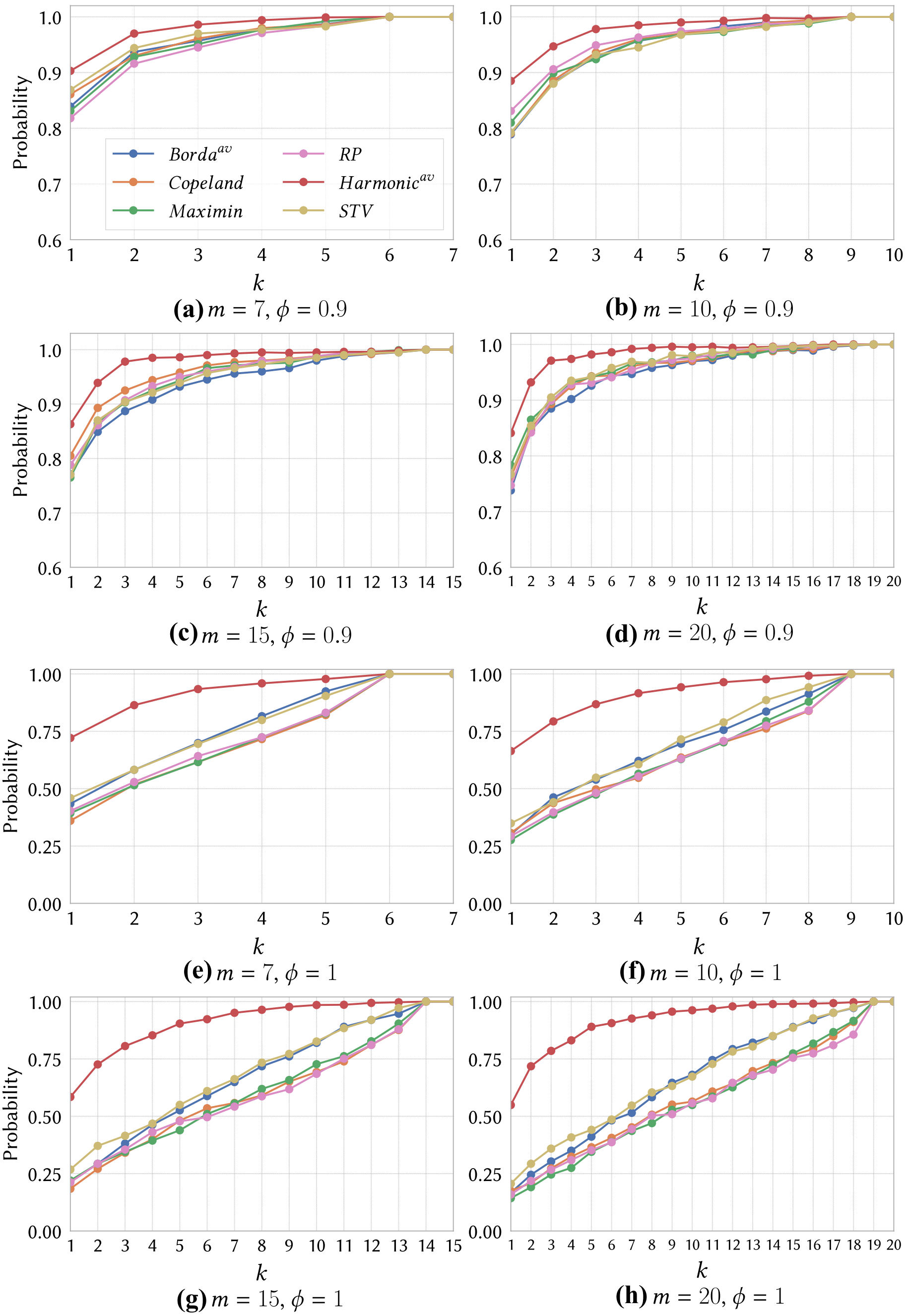
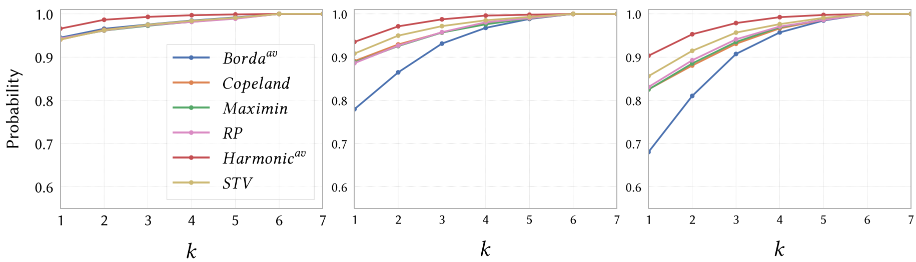


Fig. 8 Mallows, $m \in \{7, 10, 15, 20\}$, $n = 1000$, $\phi \in \{0.9, 1\}$ and varying k



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Fig. 9 Mixture of p Mallows, $m = 7$, $n = 500$, varying k and p

Figure 9 considers correlated distributions constructed by mixtures of p Mallows models when $p \in \{1, 2, 3\}$ with a fixed value of $m = 7$ and $n = 500$. We now explain in detail how we generate profiles using mixtures of Mallows models. We take $p = 3$ (the case of $p = 2$ is similar). To generate a profile:

1. We draw the reference rankings $(\sigma_1, \sigma_2, \sigma_3)$ independently, using the uniform distribution over all rankings of candidates for each.
2. We draw the dispersion parameters (ϕ_1, ϕ_2, ϕ_3) independently, using the uniform distribution over [0, 1] for each.
3. For the mixing coefficients $(\lambda_1, \lambda_2, \lambda_3)$, we generate $(x_1, x_2, x_3) \in [1, \dots, 100]$ and then we take, for each i, $\lambda_i = \frac{x_i}{x_1+x_2+x_3}$,
4. We draw each vote by first picking one of the three Mallows models with probability λ_i $(i = 1, 2, 3)$, and then generating the ranking according to (σ_i, ϕ_i).

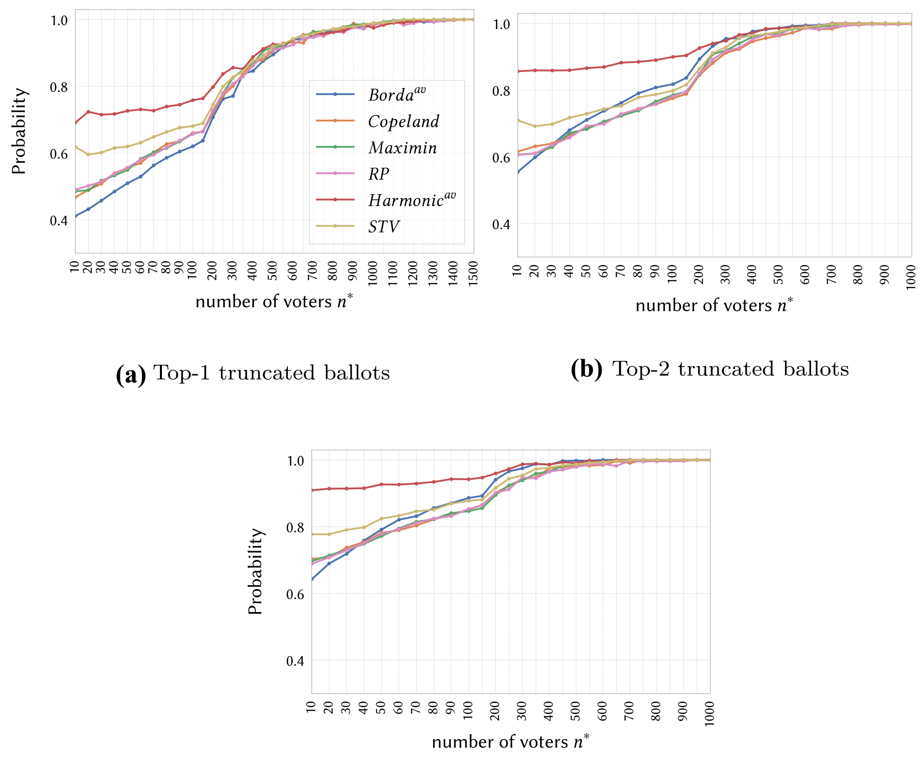
Figure 10 uses a real data set from *Preflib* [25]: the 2002 election for Dublin North constituency, for which $m = 12$ and $n = 3662$.[3] We consider data with samples of n^* voters among n $(n^* < n)$, starting by $n^* = 10$ and increment n^* in steps of 10 (resp. 50) when $n^* \leq 100$ (resp. $n^* > 100$). In each experiment, 1000 (or 10000) random profiles are generated, each of them with n^* voters; then we consider the top-k ballots obtained from these profiles, with $k = \{1, 2, 3\}$, and we compute the frequency with which the true winner is selected. Figure 10 shows results for Dublin with samples of $n^* \in \{10, \dots, 1000\}$ voters and $k = \{1, 2, 3\}$. We zoom in the x-axis when $n^* = \{10, ..., 100\}$ in order to see the behavior of different approximations with small elections.

5.2 Comparing the accuracy of various rules

We first aim to compare rules according to how easy they are to approximate by their top-k versions. Filmus and Oren [13] evaluate empirically the efficiency of top-k voting to approximate the correct winner for the Borda, Harmonic and Copeland rules,[4] they compare the resistance of these rules to truncation under the *IC* model, with a fixed number of voters. They show that the best performance is obtained with Harmonic, followed by Borda, then Copeland (see Figure 2 in [13]). We go further by comparing more rules with

[3] The original data contains 43,942 ballots and only 3662 are complete.

[4] For the Borda and Harmonic rules, they choose the average approximation, which they call *FairCutoff*.



(a) Top-1 truncated ballots

(b) Top-2 truncated ballots

(c) Top-3 truncated ballots

Fig. 10 Dublin: varying k; $n^* = \{10, \dots, 1500\}$

different distributions over votes and with real data, and by studying the behavior of the k-truncated rules with small and large elections.

Recall that plurality needs only top-1 ballots. Therefore, we may expect that the closer a voting rule f to plurality, the easier it is to approximate f from top-k ballots. This is indeed the case when comparing positional scoring rules: the Harmonic rule, whose scoring vector is closer to that of plurality than the Borda rule, performs much better. This phenomenon occurs whatever the values of the other parameters; only its amplitude varies: the superiority of Harmonic is more marked with a smaller number of voters (Figs. 4 (a) and 5 for synthetic data, and Fig. 10 for real data) and when correlation between votes is smaller (Fig. 7). For a fixed correlation between voters, it is more marked with a larger number of candidates (Fig. 8).

For $k = 1$, our results can be viewed as answering the question: *with which probability does the true winner with respect to the chosen rule coincide with the plurality winner?* Our results suggest that with highly correlated votes and a sufficient number of voters, all 1-truncated rules tend to be very close to plurality: this can be observed on Fig. 7 (a) for highly correlated synthetic votes and Fig. 10 (a) for real data.

More surprisingly, for synthetic data, the accuracy of the other considered rules except Harmonic is typically very close to that of the Borda rule. This can be observed on Figs. 4, 6 (a), and 8. In some sense, this confirms the special status of the Borda rule within the class of positional scoring rules.

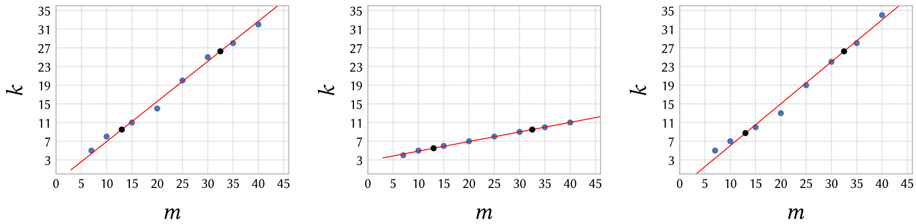
Let us focus on the set of rules {Borda, Copeland, maximin, RP, STV}. STV tends to perform slightly better than the other rules, especially when the number of voters is small (Figs. 4 (a), 5, and 10) and, to some weaker extent, when there is little or no correlation between votes (Fig. 7). This can be explained by the fact that STV gives more importance to the top positions of the votes. Copeland, maximin and RP have very similar behaviours in all settings; this may be partly explained by the fact that both rules are based on the (weighted or unweighted) majority graph, but even with this explanation we find this finding quite surprising. Finally, when there is no correlation between votes (Figs. 6 (b) and 8), STV and Borda behave in a very similar way, and better than Copeland, maximin and RP. This confirms the findings of Filmus and Oren [13] (which concerned only Harmonic, Borda and Copeland).

Results on real data (Fig. 10) confirm most of these findings: Harmonic is better, followed by STV, and then Borda, Copeland, maximin and RP, all behaving similarly (a notable difference with synthetic data is that Borda performs particular bad for $k = 1$). The superiority of STV to Borda, maximin, RP and Copeland is more marked for small values of k and n. To assess whether these numbers are statistically significant, we use Friedman's two-way analysis of variance by ranks (ANOVA by Ranks) [15], the non-parametric test that is commonly used for testing the differences between more than two related samples. (See Appendix for details.) We obtain that the null hypothesis, stating that all the rules behave similarly, is rejected at a high level of significance. Such a conclusion motivates the deep analysis carried out in the experiments.

For positional scoring rules, results with the average score are slightly better than those with the zero score (Fig. 4 (b) for the Borda rule), which is consistent with the results obtained by Filmus and Oren [13]. Moreover, such a behavior is noticeable in Table 1 where the average rank of $Borda_k^{av}$ is lower than the one of $Borda_k^0$ for the experiments in Fig. 4 (a) (the average rank of $Borda_k^{av}$ (resp. $Borda_k^0$) is 6 (resp. 6.8)). Therefore, from now on we will focus on the average-type approximations of Borda and Harmonic ($Borda_k^{av}$ and $Harmonic_k^{av}$) and most of the time we will ignore $Borda_k^0$ and $Harmonic_k^0$.

5.3 The impact of the number of votes and the correlation between them

Comparing Fig. 4 (a) and (b), as well as looking at Figs. 7 and 10, leads us to the clear conclusion that the performance of top-k approximations increases dramatically with the number of voters, everything else being equal (the rule, the correlation between votes, the value of k), except when the correlation between votes is very weak (Fig. 7 (c) and (f)). This is especially true for small values of k. By comparing Figs. 4 and 5, as well as Fig. 6 (a) and (b), Fig. 7 (a), (b) and (c), and Fig. 7 (d), (e) and (f), we observe that the accuracy of top-k approximations increases with the correlation between votes, everything else being equal. These results are consistent with those obtained by Skowron et al. [29] for *multiwinner* rules: elections with few voters and high dispersion appear to be the worst-case scenario for predicting the correct winner using top-truncated ballots. Beyond varying ϕ in the Mallows model, another way of varying the correlation between votes is to use mixtures of Mallows models. On Fig. 9 we observe that mixing more models leads to a decreased accuracy of top-k approximations (Note that increasing the number of models tend towards the impartial culture).



(a) **(b)** **(c)**

Fig. 11 Value of k needed in function of m for $n = 1000$

5.4 The elicitation-efficiency trade-off

Now we focus on the following question: depending on the voting rule used, the number of voters, the number of candidates, and the type of distribution over profiles, what is a reasonable value of k to choose? Choosing a low value usually comes with a larger risk or error, while choosing a high value induces too large a burden on the voters. A risk-averse strategy consists in identifying the smallest value of k so that *none of our generated profiles lead to an error*. Of course, this value depends on n, m, the distribution and the voting rule. As an example, let us focus on Mallows distributions, $m \in \{7, 10, 15, 20\}$, $n = 1000$.

- For $\phi = 1$, on Fig. 8 we observe that the minimal value of k is always $m - 1$, whatever the voting rule (in other words, we always find a generated profile for which we get an incorrect result if the profile is not complete).
- For $\phi = 0.9$, Fig. 11 (a) depicts the results obtained on Fig. 8 (a), (b), (c), and (d) for Borda (and also STV, Copeland, RP and maximin, for which the obtained values are the same as for Borda; Harmonic is the only exception). We have added results for $m \in \{25, 30, 35, 40\}$ to obtain a more interesting pattern. The x-axis corresponds to m and the y-axis to the value of k needed in order to get the correct winner. For instance, on Fig. 8 (c) we see that for $m = 15$, $k = 11$ is needed. The value of k needed as a function of m follows a surprisingly linear pattern.[5] (the red line in Fig. 11 (a)). The equation associated to this line is $\frac{17}{20}m - \frac{8}{5}$.
- For $\phi = 0.8$, $k = 2$ is always sufficient whatever the value of m.
- For $\phi = 0.7$, $k = 1$ is always sufficient, whatever m.

For Harmonic, we observe on Fig. 8 that $k = 1$ is always sufficient for $\phi \leq 0.8$ and $n = 1000$. For $\phi = 0.9$ (resp. $\phi = 1$), Fig. 11 (b) (resp. Fig. 11 (c)) depicts the results obtained on Fig. 8 (a), (b), (c), and (d) (resp. Fig. 8 (e), (f), (g), and (h)); the equation associated to the drawn line is $\frac{1}{5}m + \frac{14}{5}$ (resp. $\frac{35}{39}m - \frac{11}{12}$) which corresponds to the value of k needed.

With real data, on Fig. 10 we can observe an interesting trade-off between the number of voters n^* and the amount of information to elicit form them for obtaining correct winner

[5] It consists of dividing a distribution of data – arranged in the increasing order of their abscissas – into two subgroups of equal size and then calculating a mean point for each of them (the black points in Fig. 11). We draw the line that joins these two points. This line passes through the centre of the scatter plot.

selection with 100% accuracy: for all rules, $k = 1$ is sufficient to predict the correct winner if $n^* \geq 1400$; for $k = 2$, $n^* \geq 850$; and for $k = 3$, $n^* \geq 750$.[6]

6 Measuring the approximation ratio

In the previous section, we measured empirically the quality of the approximation by the frequency with which it outputs the true winner of the original rule. Now, we follow the second path: we evaluate the quality of the approximations by computing the score ratio between the score of the true winner and the winner of the approximate rule. In Sect. 6.1 we analyze theoretically the score distortion in the worst case. We complete our study by an empirical analysis of the average-case study and a study from real data sets in Sect. 6.2 and Sect. 6.3, respectively. We will see that average-case and real-case experiments perform much better than the worst case.

6.1 Worst case study

In order to measure the quality of approximate voting rules whose definition is based on score maximization, a classical method consists in computing the worst-case approximation ratio between the scores (with respect to the original rule) of the "true" winner and of the winner of the approximate rule. It is interesting to note that the performance of $Borda_k^{av}$ and $Borda_k^0$ is highly similar (it is very slightly better for $Borda_k^{av}$).

Using worst-case score ratios is classical: they are defined for measuring the quality of approximate voting rules [7, 28], for defining the price of anarchy of a voting rule [6] or for measuring the distortion of a voting rule [4]. In this context, Bentert and Skowron [3] consider top-k approximations of positional scoring rules and the maximin rule, that take into account k-truncated ballots. They give tight bounds for all positional scoring rules and for maximin. We will discuss them in more detail in the subsections where we consider these rules.

Worst-case score ratios particularly make sense if the score of a candidate is meaningful beyond its use for determining the winner. This is definitely the case for Borda, as the Borda count is often seen as a measure of social welfare (see [9]); this is also the case, to a slightly weaker extent, for other positional scoring rules (including Harmonic); the Copeland and maximin scores are arguably less meaningful as measures of social welfare. This worst-case score ratio is called the *price of top-k truncation*.

Definition 4 Let f be a voting rule defined as the maximization of a score S, and f_k a top-k approximation of f. The *price of top-k-truncation* for f, f_k, m, and k, is defined as:

$$R(f, f_k, m, k) = \max_{P \in \mathcal{P}_m} \frac{S(f(P))}{S(f_k(P_k))}$$

[6] Thus we obtain the correct result with 1400 voters specifying their top candidate (which needs $1400 \log 12 \approx 3479$ bits to be communicated), or 850 voters specifying their top-2 candidates (which needs $850 \log(12 \times 11) \approx 4150$ bits), or 750 voters specifying their top-3 candidates (which needs $750 \log(12 \times 11 \times 10) \approx 5389$ bits).

6.1.1 Positional scoring rules

Let f^s be a positional scoring rule defined with scoring vector s. Assume the tie-breaking priority favors x_1. Let $f_k^{\bar{s}}$ be a top-k approximation of f^s, associated with vector $\bar{s} = (s_1, \ldots, s_k, s^*)$, with the same tie-breaking priority. Let $s' = (s_1 - s^*, \ldots, s_k - s^*, 0) = (s'_1, \ldots, s'_k, 0)$, i.e., $s'_i = s_i - s^*$ for $i = 1, \ldots, m$. Obviously, $f_k^{\bar{s}} = f_k^{s'}$. For instance, if $f^{\bar{s}}$ is the average-score approximation of the Borda rule, then $\bar{s} = (m-1, \ldots, m-k, \frac{m-k-1}{2})$ and $s' = (m-1-\frac{m-k-1}{2}, \ldots, m-k-\frac{m-k-1}{2}, 0)$.

Let $S(x, P)$ be the score of x for P under f^s and $S'_k(x, P_k)$ be the score of x for P_k under $f_k^{s'}$. From now on when we write scores we omit P and P_k, i.e., we write $S(x)$ instead of $S(x, P)$, $S'_k(x)$ instead of $S'_k(x, P_k)$ etc. In the rest of Sect. 6.1 we assume $k \geq 2$. Let $x_1 = f_k^{s'}(P_k)$ and $x_2 = f^s(P)$.

Let α (resp. β) be the number of ballots in which x_1 (resp. x_2) appears in the top k positions. Also, let us write $S(x) = S_{1\to k}(x) + S_{k+1\to m}(x)$, where $S_{1\to k}(x)$ (resp. $S_{k+1\to m}(x)$) is the number of points that x gets from the top k (resp. bottom $m-k$) positions of the ballots in P. Let us denote $\Sigma' = s'_1 + \ldots + s'_k$.

Lemma 2 $S'_k(x_1) \geq \frac{n}{m}\Sigma'$

Proof The total number of points given to candidates under $f_k^{s'}$ is $n\Sigma'$, therefore $S'_k(x_1) \geq \frac{n}{m}\Sigma'$.

Lemma 3 $\alpha \geq \frac{n}{ms'_1}\Sigma'$

Proof $S'_k(x_1) \leq \alpha s'_1$, therefore, from Lemma 2,

$$\alpha \geq \frac{S'_k(x_1)}{s'_1} \geq \frac{n}{ms'_1}\Sigma'$$

□

From Lemmas 2 and 3 we have:

$$S(x_1) \geq \frac{n\Sigma'}{m} + \alpha s^* \geq \frac{n\Sigma'}{m} + \frac{n\Sigma' s^*}{ms'_1} = \frac{n\Sigma'}{m}\left(1 + \frac{s^*}{s'_1}\right)$$

Now we consider separately the cases $s^* = 0$ and $s^* > 0$, since in the former case we get a tight bound that we do not get in the latter.

Lemma 4 *If* $s^* = 0$ *then* $R(f^s, f_k^{s'}, m, k) \leq 1 - \frac{s_{k+1}}{s_1} + \frac{ms_{k+1}}{s_1 + \ldots + s_k}$

Proof Since $s^* = 0$ we have $s'_i = s_i$ for all i and $S'_k = S_k$; therefore $S_{1\to k}(x_1) = S'_k(x_1) + \alpha s^* = S_k(x_1)$, and $S(x_1) \geq \frac{n}{m}(s_1 + \ldots + s_k)$. As x_2 appears in at least $\frac{S_k(x_2)}{s_1}$ top-k ballots, we have $\beta \geq \frac{S_k(x_2)}{s_1}$. Moreover we have $S(x_1) \geq S_{1\to k}(x_1) = S_k(x_1) \geq S_k(x_2) = S_{1\to k}(x_2)$. Now,

$$
\begin{array}{rcl}
S(x_2) & \leq & S_{1\to k}(x_2) + \left(n - \frac{S_k(x_2)}{s_1}\right)s_{k+1} \\
& = & (1 - \frac{s_{k+1}}{s_1})S_{1\to k}(x_2) + ns_{k+1} \\
& \leq & (1 - \frac{s_{k+1}}{s_1})S(x_1) + ns_{k+1} \\
\frac{S(x_2)}{S(x_1)} & \leq & 1 - \frac{s_{k+1}}{s_1} + ns_{k+1}\frac{m}{n(s_1+\ldots+s_k)} \\
& = & 1 - \frac{s_{k+1}}{s_1} + \frac{ms_{k+1}}{s_1+\ldots+s_k}
\end{array}
$$

□

Lemma 5 $R(f^s, f_k^{s'}, m, k) \leq 1 + \frac{ms'_1 s_{k+1}}{s_1\Sigma'} - \frac{s^*}{s_1}$

Proof We have the following inequalities:

$$
\begin{array}{rcl}
S(x_2) & \leq & S_{1\to k}(x_2) + (n-\beta)s_{k+1} \\
& = & S'_k(x_2) + \beta s^* + (n-\beta)s_{k+1} \\
& \leq & S'_k(x_1) + \beta s^* + (n-\beta)s_{k+1} \ \ \text{(because } x_1 = f_k^{s'}(P_k)\text{)} \\
& = & S_{1\to k}(x_1) - \alpha s^* + \beta s^* + (n-\beta)s_{k+1} \\
& \leq & S(x_1) - \beta(s_{k+1} - s^*) - \alpha s^* + ns_{k+1} \\
& \leq & S(x_1) - \alpha s^* + ns_{k+1} \ \ \text{(because } s_{k+1} \geq s^*\text{)}
\end{array}
$$

Since $S(x_1) \geq \frac{n\Sigma'}{m} + \alpha s^*$, we get:

$$\frac{S(x_2)}{S(x_1)} \leq 1 + \frac{ns_{k+1} - \alpha s^*}{\frac{n\Sigma'}{m} + \alpha s^*} = 1 + \frac{m(ns_{k+1} - \alpha s^*)}{n\Sigma' + m\alpha s^*}$$

From Lemma 3, $\alpha s^* \geq \frac{n\Sigma' s^*}{ms'_1}$, which gives us

$$
\begin{array}{rcl}
\frac{S(x_2)}{S(x_1)} & \leq & 1 + \frac{m}{n\Sigma' + \frac{n\Sigma' s^*}{s'_1}}\left(ns_{k+1} - \frac{n\Sigma' s^*}{ms'_1}\right) \\
& = & 1 + \frac{m}{\Sigma' + \frac{\Sigma' s^*}{s'_1}}\left(s_{k+1} - \frac{\Sigma' s^*}{ms'_1}\right) \\
& = & 1 + \frac{ms'_1}{s'_1\Sigma' + \Sigma' s^*}\frac{ms'_1 s_{k+1} - \Sigma' s^*}{ms'_1} \\
& = & 1 + \frac{ms'_1 s_{k+1} - \Sigma' s^*}{(s'_1 + s^*)\Sigma'} \\
& = & 1 + \frac{ms'_1 s_{k+1} - \Sigma' s^*}{s_1\Sigma'} \\
& = & 1 + \frac{ms'_1 s_{k+1}}{s_1\Sigma'} - \frac{s^*}{s_1}
\end{array}
$$

□

If we apply this bound to the special case $s^* = 0$, we get $\frac{S(x_2)}{S(x_1)} \leq 1 + \frac{ms_1 s_{k+1}}{s_1(s_1+\ldots+s_k)} = 1 + \frac{ms_{k+1}}{s_1+\ldots+s_k}$: comparing it to the bound $1 - \frac{s_{k+1}}{s_1} + \frac{ms_{k+1}}{s_1+\ldots+s_k}$ of Lemma 4, we see that it is lower. This already enables us to say that the bound of Lemma 5 will not be tight (we will see that on the other hand, the bound of Lemma 4 is tight).

We now focus on the lower bound. We build the following pathological complete profile P such that:

– the winner for P_k (resp. P) is x_1 (resp. x_2).

- In P_k, all candidates get the same number of points (x_1 wins thanks to tie-breaking), and x_1 and x_2 get all their points from top-1 positions.
- In P, the score of x_1 is minimized by ranking it last everywhere where it was not in the top k positions, and the score of x_2 is maximized by ranking it in position $k+1$ everywhere where it was not in the top k positions.
- P_k is symmetric in $\{x_3, \ldots, x_m\}$.

Formally, P_k is defined as follows:

1. For each ranked list L (resp. L') of $k-1$ (resp. k) candidates in $\{x_3, \ldots, x_m\}$: α votes $x_1 L$ and α votes $x_2 L$ (resp. β votes L'). α and β will be fixed later.
2. α and β are chosen in such a way that all candidates get the same score $S'_k(.)$.

Now, P is obtained by completing P_k as follows:

1. Each top-k vote $x_1 L$ is completed into $x_1 L x_2 -$. "$-$" means the remaining candidates are in an arbitrary order.
2. Each top-k vote $x_2 L$ is completed into $x_2 L - x_1$.
3. Each top-k vote L' is completed into $L' x_2 - x_1$.

For instance, for $m = 5$ and $k = 3$, P is as follows:

$\alpha\ x_1x_3x_4x_2x_5$	$\alpha\ x_2x_3x_4x_5x_1$	$\beta\ x_3x_4x_5x_2x_1$
$\alpha\ x_1x_3x_5x_2x_4$	$\alpha\ x_2x_3x_5x_4x_1$	$\beta\ x_3x_5x_4x_2x_1$
$\alpha\ x_1x_4x_3x_2x_5$	$\alpha\ x_2x_4x_3x_5x_1$	$\beta\ x_4x_3x_5x_2x_1$
$\alpha\ x_1x_4x_5x_2x_3$	$\alpha\ x_2x_4x_5x_3x_1$	$\beta\ x_4x_5x_3x_2x_1$
$\alpha\ x_1x_5x_3x_2x_4$	$\alpha\ x_2x_5x_3x_4x_1$	$\beta\ x_5x_3x_4x_2x_1$
$\alpha\ x_1x_5x_4x_2x_3$	$\alpha\ x_2x_5x_4x_3x_1$	$\beta\ x_5x_4x_3x_2x_1$

Let $M = \frac{(m-3)!}{(m-k-1)!}$ and $Q = \frac{(m-2)!}{(m-k-1)!}$.

Lemma 6

$$S'_k(x_1) = S'_k(x_2) = \alpha(m-2)s'_1 M$$

and for $i \geq 3$, $S'_k(x_i) = 2\alpha(s'_2 + \ldots + s'_k)M + \beta(m-k-1)(s'_1 + \ldots + s'_k)M$

Proof In P_k, x_1 and x_2 appear in top position in a number of votes equal to α times the number of different permutations (ordered lists) of $(k-1)$ candidates out of $(m-2)$, i.e. $\alpha\frac{(m-2)!}{(m-k-1)!}$ times. Thus $S'_k(x_1) = S'_k(x_2) = \alpha\frac{(m-2)!}{(m-k-1)!}s'_1$. For similar reasons, for each $i \geq 3$,

$$S'_k(x_i) = 2\alpha\frac{(m-3)!}{(m-k-1)!}(s'_2 + \cdots + s'_k) + \beta\frac{(m-3)!}{(m-k-2)!}(s'_1 + \cdots + s'_k).$$

$\square$

As a consequence, all candidates have the same score in P_k if and only if

$$\frac{\beta}{\alpha} = \frac{(m-2)s'_1 - 2(s'_2 + \ldots + s'_k)}{(m-k-1)(s'_1 + \ldots + s'_k)}$$

We fix α and β such that this equality holds. Thanks to the tie-breaking priority, the winner in P_k is x_1. In P, the winner is x_2 and the scores of x_1 and x_2 are as follows:

$$\begin{aligned} S(x_1) &= Q\alpha s_1 \\ S(x_2) &= Q\alpha s_1 + Q\alpha s_{k+1} + Q(m-k-1)\beta s_{k+1} \end{aligned}$$

Lemma 7

Proof x_1 appears at the top of $\frac{(m-2)!}{(m-k-1)!}\alpha$ votes and at the bottom of all others, hence $S(x_1) = Q\alpha s_1$. x_2 appears $\alpha\frac{(m-2)!}{(m-k-1)!}$ times top position, and in position $(k+1)$ in the remaining votes, *i.e.*, $\alpha\frac{(m-2)!}{(m-k-1)!} + \beta\frac{(m-2)!}{(m-k-2)!}$. Thus

$$S(x_2) = \alpha\frac{(m-2)!}{(m-k-1)!}(s_1 + s_{k+1}) + \beta\frac{(m-2)!}{(m-k-2)!}s_{k+1}$$

□

Lemma 8 $R(f^s, f_k^{s'}, m, k) \geq 1 - \frac{s_{k+1}}{s_1} + \frac{s_{k+1}}{s_1}\frac{ms'_1}{s'_1+\ldots+s'_k}$

Proof From Lemma 7 we get $\frac{S(x_2)}{S(x_1)} \geq 1 + \frac{s_{k+1}}{s_1} + (m-k-1)\frac{s_{k+1}}{s_1}\frac{\beta}{\alpha}$.

Finally, using the expression of $\frac{\beta}{\alpha}$ we get

$$\frac{S(x_2)}{S(x_1)} \geq 1 + \frac{s_{k+1}}{s_1} + (m-k-1)\frac{s_{k+1}}{s_1}\frac{(m-2)s'_1 - 2(s'_2+\ldots+s'_k)}{(m-k-1)(s'_1+\ldots+s'_k)}$$

From this we conclude:

$$\begin{aligned} R(f^s, f_k^{s'}, m, k) &\geq 1 + \frac{s_{k+1}}{s_1} + \frac{s_{k+1}}{s_1}\frac{(m-2)s'_1 - 2(s'_2+\ldots+s'_k)}{s'_1+\ldots+s'_k} \\ &= 1 + \frac{s_{k+1}}{s_1} + \frac{s_{k+1}}{s_1}\frac{(m-2)s'_1 + 2s'_1 - 2(s'_1+\ldots+s'_k)}{s'_1+\ldots+s'_k} \\ &= 1 + \frac{s_{k+1}}{s_1} + \frac{s_{k+1}}{s_1}\left(\frac{ms'_1}{s'_1+\ldots+s'_k} - 2\right) \\ &= 1 - \frac{s_{k+1}}{s_1} + \frac{s_{k+1}}{s_1}\frac{ms'_1}{s'_1+\ldots+s'_k} \end{aligned}$$

□

Putting Lemmas 5 and 8 together we get

$$1 - \frac{s_{k+1}}{s_1} + \frac{s_{k+1}}{s_1}\frac{ms'_1}{s'_1+\ldots+s'_k} \leq R(f^s, f_k^{s'}, m, k) \leq 1 - \frac{s^*}{s_1} + \frac{s_{k+1}}{s_1}\frac{ms'_1}{s'_1+\ldots+s'_k}$$

Proposition 1

There is a small gap between the two bounds. Now let us consider the case $s^* = 0$ apart: putting Lemmas 4 and 8 together we get

Proposition 2 *If* $s^* = 0$ *then* $R(f^s, f_k^{s'}, m, k) = 1 - \frac{s_{k+1}}{s_1} + \frac{ms_{k+1}}{s_1+\ldots+s_k}$

So when $s^* = 0$ we have a tight worst-case approximation ratio. Moreover, our (lower and upper) bound coincides with the optimal ratio given in [3] (Theorem 1).[7] Since the ratio in [3] is shown to be the best possible ratio, this show that taking $s^* = 0$ gives an optimal top-k approximation of a positional scoring rule.[8]

In particular:

[7] Note that the ratios in our paper are the inverse of the ratios in [3]. That is, the inverse of the ratio given in Theorem 1 of [3] coincides with our ratio for $s^* = 0$.

[8] Interestingly, [3] give another optimal rule (thus with same worst-case ratio), which is much more complex, and which is not a top-k PSR. Comparing the average ratio of both rules is left for further study.

- For $Borda_k^0$ ($s_i = m - i, s^* = 0$), the lower and upper bounds coincide and are equal to $\frac{k}{m-1} + \frac{2m(m-k-1)}{k(2m-k-1)}$.
- for $Borda_k^{av}$ ($s_i = m - i, s^* = \frac{m-k-1}{2}$), the lower bound is $1 - \frac{m-k-1}{m-1} + \frac{(m-k-1)(m+k-1)}{k(m-1)}$ and the upper bound is $1 + \frac{(m-k-1)(m+k-1)}{k(m-1)} - \frac{m-k-1}{2(m-1)}$.
- for $Harmonic_k^0$ ($s_i = \frac{1}{i}, s^* = 0$), the lower and upper bounds are equal to $\frac{k}{k+1} + \frac{m}{(k+1)(1+\frac{1}{2}\cdots+\frac{1}{k})}$.

Also, note that for k'-approval with $k' > k$ and $s^* = 0$, the (exact) worst-case ratio $\frac{m}{k}$ does not depend on k'. As a corollary, we get the following order of agnitudes when m grows:

- $R(Borda, Borda_k^0, m, k) = \Theta\left(\frac{m}{k}\right)$.
- $R(Borda, Borda_k^{av}, m, k) = \Theta\left(\frac{m}{k}\right)$.
- $R(Harmonic, Harmonic_k^0, m, k) = \Theta\left(1 + \frac{m}{k \log k}\right)$.

6.1.2 maximin

We now consider the *maximin* rule, with tie-breaking priority $x_1 \dots x_m$, and $maximin_k$ its the k-truncated version with the same tie-breaking priority. Let $S_{Mm}(x_2, P)$ and $S_{Mm}(x_1, P_k)$ be the maximin scores of x_2 and x_1 for P and P_k, respectively, with $S_{Mm}(x_2, P) = \min_{y \neq x_2} N_P(x_2, y)$ and similarly for P_k. Let P be a profile, and let $x_1 = maximin_k(P_k)$ and $x_2 = maximin(P)$. All candidates have the same maximin score in P_k, therefore, by tie-breaking priority, $maximin_k(P_k) = x_1$.

Lemma 9 $R(maximin, maximin_k, m, k) \leq m - k + 1$.

Proof Because $x_1 = maximin_k(P_k)$, we must have $S_{Mm}(x_1, P_k) \geq 1$ (otherwise we would have $S_{Mm}(x_1, P_k) \geq 0$, meaning that x_1 does not belong to any top-k ballot, and in this case we cannot have $x_1 = maximin_k(P_k)$). Now, $S_{Mm}(x_2, P) \leq S_{Mm}(x_2, P_k) + (m - k) \leq S_{Mm}(x_1, P_k) + (m - k)$, therefore,

$$\begin{aligned} \frac{S_{Mm}(x_2,P)}{S_{Mm}(x_1,P)} &\leq \frac{S_{Mm}(x_1,P_k)+(m-k)}{S_{Mm}(x_1,P_k)} \\ &\leq m - k + 1 \end{aligned}$$

□

Lemma 10 $R(maximin, maximin_k, m, k) \geq m - k$.

Proof We consider the cyclic profile Cyc:

Cyc					$P(m=5, k=2)$				
x_1	x_2	$\dots$	$m-1$	m	x_1	x_2	x_3	x_4	x_5
x_2	x_3	$\dots$	m	x_1	x_2	x_3	x_4	x_5	x_1
x_3	x_4	$\dots$	x_1	x_2	x_3	x_4	x_2	x_5	x_1
$\dots$	$\dots$	$\dots$	$\dots$		x_4	x_5	x_2	x_3	x_1
m	x_1	$\dots$	$m-2$	$m-1$	x_5	x_1	x_2	x_3	x_4

Now, let P be obtained from Cyc by the following operations for every vote in Cyc:

- If x_1 is not in the top k positions in the vote, we move it to the last position (and move all candidates who were below x_1 one position upward)
- If x_2 is not in the top k positions in the vote, we move it to the $(k+1)^{th}$ position (and move all candidates who were between position $k+1$ and 2's position one position downward).

For instance, for $m = 5$, $k = 2$, we get the profile P above.

$maximin(P) = x_2$, and the maximin scores of x_1 and x_2 in P are:

$$S_{Mm}(x_1, P) = 1 \text{ and } S_{Mm}(x_2, P) = m - k.$$

Hence $\frac{S_{Mm}(x_2,P)}{S_{Mm}(x_1,P)} = m - k.$ □

Proposition 3 $m - k \leq R(maximin, maximin_k, m, k) \leq m - k + 1.$

This worst-case ratio is quite bad, except if k is close to m. However, arguably, the maximin score makes less sense *per se* (i.e., as a measure of social welfare) than a positional score such as the Borda count. The obtained lower bound matches the one given by Bentert and Skowron [3] (Section 4.3) which means that our top-k approximation of maximin is optimal.

6.1.3 Copeland

Again, for the Copeland rule, the ratio makes less sense, because the Copeland score is less meaningful as a measure of social welfare.[9] Still, for the sake of completeness we give the following result:

Proposition 4 $R(Copeland, Copeland_k, m, k) = \infty.$

Proof Let P be the following profile:

- P_k contains two votes $x_1 x_2 \dots x_k$, and one vote L for each ordered list of k candidates among m.
- P is obtained by completing P_k by adding x_1 (resp. x_2) in last position (resp. in position $k+1$) when it is not in the *top-k* positions.

In P_k, the winner for $Copeland_k$ is x_1. In P, the Copeland winner is x_2. Now, with respect to P, the Copeland score of x_1 (resp. x_2) is 0 (resp. $m-1$), hence the result. □

6.1.4 Discussion

The price of truncation is very diverse across different voting rules. For positional scoring rules, especially Harmonic, it is reasonable, especially if k is not too small; for maximin

[9] Moreover, there are several ways of defining the Copeland score, all leading to the same rule. However, this has no impact on the negative result below, as long as a Condorcet loser has score 0.

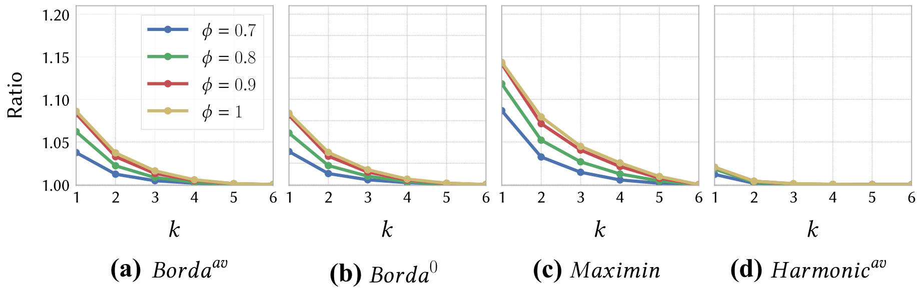


Fig. 12 Mallows: Approximation ratio for $n = 15$, $m = 7$ and varying ϕ

and Copeland, it is much less good, but this has to be tempered by the fact that, as we said, the scores for these two rules make less sense as a measure of social welfare than positional scores.[10] For instance, if $k = \frac{m}{4}$ (which means that voters have to report one fourth of their rankings), the approximation ratios for Harmonic, Borda, maximin and Copeland are close to, respectively, $1 + \frac{4}{log(\frac{m}{4})}$, 4, $\frac{3m}{4}$ and ∞.

Now, we may wonder whether these worst cases do occur frequently in practice or if they correspond to rare pathological profiles. The next two subsections show that the latter is the case.

6.2 Average case evaluation

We present the evaluation of the approximation ratio for different truncated rules using data generated from Mallows and mixtures of Mallows. We will focus on small elections since the accuracy is good with large values of n. For each experiment, we draw 10000 random profiles, with $m = 7$, $n = 15$, and let ϕ and p vary. Figure 12 shows obtained results with Mallows with $\phi \in \{.7, .8, .9, 1\}$. Figure 13 reports on results with mixtures of p Mallows with $p = \{1, 2, 3\}$.[11]

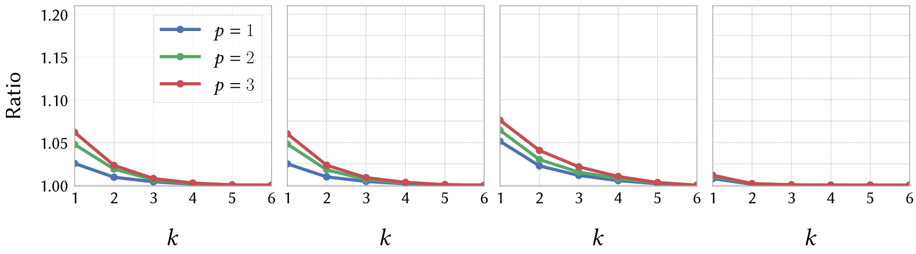
In the same context, Bentert and Skowron [3] measure empirically the approximation ratio for Borda (with the average-score version) and maximin. With a small number of voters, their results show that the quality of the approximation increases with the correlation between votes. Our results confirm and complement theirs. For large ϕ and p, the approximation ratio increases except for the Harmonic rule, for which the approximation ratio is very close to 1 for all values of ϕ and p (see Figs. 12 (d) and 13 (d)). In general, when the number of voters is small, a better quality approximation is obtained from mixtures of Mallows than with Mallows models, which is consistent with the results in [3] (see Figs. 12 (c) and 13 (c)).

It is interesting to note that the performance of $Borda_k^{av}$ and $Borda_k^0$ is highly similar (it is very slightly better for $Borda_k^{av}$).

Our empirical results are much, much better than the worst case; however, the relative performance of rules is similar to what we obtained for the worst case: the best result is obtained by Harmonic, followed by Borda and finally maximin.

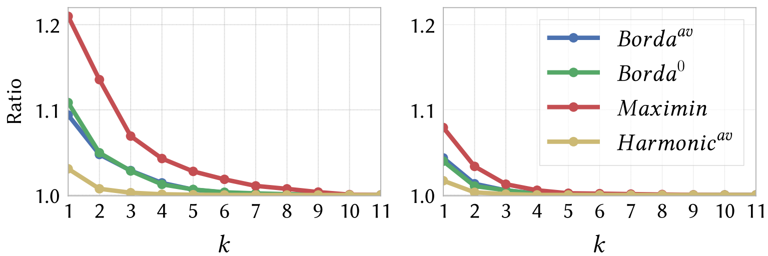
10 As Ranked Pairs is not based on scores, it was not studied here. All others rules we considered, including Copeland and maximin, are defined via a score maximization.

11 For the method used for generating mixtures of Mallows model see the discussion page 11.



(a) $Borda^{av}$ **(b)** $Borda^{0}$ **(c)** $Maximin$ **(d)** $Harmonic^{av}$

Fig. 13 Mixtures of p Mallows: Approximation ratio for $n = 15$, $m = 7$ and $p \in \{1, 2, 3\}$



(a) $n = 15$ **(b)** $n = 100$

Fig. 14 Approximation ratio with Dublin North data set

6.3 Real data sets

Again we consider 2002 Dublin North data ($m = 12, n = 3662$) with samples of n^* voters among n ($n^* < n$), with $n^* = \{15, 100\}$ (see Fig. 14). In each experiment 10000 random profiles are constructed with n^* voters; then we consider the top-k ballots obtained from these profiles with $k = \{1, \ldots, m-1\}$. Again, the results are very positive.

7 Conclusion

In this paper we have considered k-truncated approximations of rules (Borda, Harmonic, Copeland, maximin and ranked pairs), taking *top-k* ballots as input. We have considered two measures of the quality of the approximation: the probability of selecting the same winner as the original rule, and the score ratio.

For the first measure, we have measured empirically the quality of the k-truncated rules by the frequency with which they output the true winner. Empirical results, based on randomly generated profiles and on real data, demonstrate the practical viability and advantages of our approximations. Our empirical study suggest that a very small k suffices.

For the second measure, we have studied the theoretical bounds of these approximations, for rules whose definition is based on score maximization (Borda, Harmonic, Copeland and maximin), by identifying the order of the worst-case ratio. Also, we have tested the ability of these truncated rules to predict the standard voting rules based on both randomly generated profiles and real data. While the theoretical bounds are, at best, moderately encouraging, our experiments show that in practice the approximation ratio is much better than in the worst case: our results suggest that a very small value of k works very well in practice.

This paper focuses on the approximation of voting rules by top-k ballots and evaluates them *only in that respect*. This is only one criteria out of many, and we certainly do not want the reader to believe that rules that are easy to approximate from top-k ballots are better that those that are not.

Many issues remain open. Especially, it would be interesting to consider top-k approximations as voting rules on their own, and to study their normative properties.

Appendix: ANOVA by Ranks

The Friedman test [15] known as the Friedman two-way ANalysis Of VAriances by ranks (*ANOVA by Ranks*) is a non-parametric statistical test. The objective of this test is to determine if we may conclude from a sample of results that there is difference among treatment effects:

$$\chi_F^2 = \frac{12b}{v(v+1)} \sum_{j=1..v} R_j^2 - \frac{v \times (v+1)^2}{4} \tag{1}$$

where

- b is number of data sets (blocks, rows);
- v is number of treatments (voting rules, columns);
- r_i^j is the rank of the *jth* of v rules on the *ith* of b data sets;
- $R_j = \frac{1}{b} \sum_{i=1..b} r_i^j$

The first step in calculating the ANOVA's test is to convert the original results to ranks. Thus, it ranks the treatments for each problem separately, the best performing treatment should have rank 1, the second best rank 2, etc. In case of ties, average ranks are computed.

Under the null hypothesis (H_0) — which states that all the treatments behave similarly and thus their ranks R_j for $j = \{1, \dots, v\}$ should be equal — the Friedman statistic is distributed according to χ_F^2 with $b - 1$ degrees of freedom when b and v are big enough (as a rule of a thumb, $b > 10$ and $v > 5$).

Iman and Davenport [17] showed that Friedman's χ_F^2 presents a conservative behavior and proposed a better statistic which is distributed according to the F-distribution with two degrees of freedom $v - 1$ and $(v - 1) \times (b - 1)$:

$$F_F = \frac{(b-1) \times \chi_F^2}{b \times (v-1) - \chi_F^2} \tag{2}$$

In order to apply the ANOVA's test to our specific case, let us consider the experimental results illustrated in Fig. 4 (a) when $m = 7, n = 15$ and $\phi = 0.8$ presented in Table 1.

Our goal is to compare statistically the behavior of voting rules (columns) for different values of $k \in \{1, \dots, 5\}$ (rows). In this example, $b = 5$ and $v = 7$. We compute χ_F^2 and F_F following Eqs. 1 and 2, respectively as follows:

Table 1 ANOVA's test for experiments in Fig. 4 (a)

	Copeland	Maximin	Borda-av	Borda-0	RP	Harmonic	STV
k = 1	0.611 (4)	0.6107 (5)	0.5818 (6)	0.5811 (7)	0.6137 (3)	0.7772 (1)	0.6521 (2)
k = 2	0.7405 (4)	0.742 (3)	0.7228 (7)	0.7292 (6)	0.7384 (5)	0.9062 (1)	0.7748 (2)
k = 3	0.8297 (4)	0.832 (3)	0.8231 (6)	0.8184 (7)	0.8233 (5)	0.955 (1)	0.8699 (2)
k = 4	0.8992 (4)	0.9008 (3)	0.8972 (5)	0.8876 (7)	0.8947 (6)	0.9805 (1)	0.9369 (2)
k = 5	0.9629 (3)	0.9601 (4)	0.9525 (6)	0.9432 (7)	0.9592 (5)	0.993 (1)	0.9826 (2)
Avg. Rank	3.8	3.6	6	6.8	4.8	1	2

$$\chi_F^2 = \frac{12 \times 5}{7 \times 8}[(3.8^2 + 3.6^2 + +6^2 + 6.8^2 + 4.8^2 + 1^2 + 2^2) - \frac{7 \times (8)^2}{4}] = 27.514$$

$$F_F = \frac{4 \times 27.514}{5 \times 6 - 27.514} = 44.27$$

The Friedman test proves whether the measured average ranks are significantly different from the mean rank $R_j = (3.8 + 3.6 + 6 + 6.8 + 4.8 + 1 + 2)/7 = 4$ where $j \in \{1, \dots, 7\}$ expected under the null hypothesis.

F_F is distributed according to the F distribution with $DF1 = 7 - 1 = 6$ and $DF2 = (7 - 1) \times (5 - 1) = 24$ degrees of freedom. In this example, the null hypothesis is rejected because:

Table 2 Statistical test for different experiments

Figure	χ_F^2	F_F	DF1	DF2	Critical value	*p-value*	H_0
Figure 4 (a)	27.514	44.2758	6	24	2.50818	8.12E-12	REJECT
Figure 4 (b)	20,20714	8,25382	6	24	2.50818	6,51E-05	REJECT
Figure 5 (a)	19.85714	15.44444	5	20	2.71089	2.82E-06	REJECT
Figure 5 (b)	20.2	16.83333	5	20	2.71089	1.45E-06	REJECT
Figure 6 (a)	66.98413	49.47586	5	85	2.321812	9.58E-24	REJECT
Figure 6 (b)	79.01587	122.2919	5	85	2.321812	2.71E-37	REJECT
Figure 8 (a)	17.6	9.513514	5	20	2.71089	9.16E-05	REJECT
Figure 8 (b)	27.08929	14.68741	5	35	2.485143	9.00E-08	REJECT
Figure 8 (c)	48.01099	33.91203	5	60	2.36827	2.79E-16	REJECT
Figure 8 (d)	67.07143	49.72897	5	85	2.321812	8.17E-24	REJECT
Figure 8 (e)	22.25714	32.45833	5	20	2.71089	6.12E-09	REJECT
Figure 8 (f)	35.21429	51.50746	5	35	2.485143	3.72E-15	REJECT
Figure 8 (g)	55.23077	67.84252	5	60	2.36827	2.11E-23	REJECT
Figure 8 (h)	77.56349	106.0249	5	85	2.321812	5.16E-35	REJECT
Figure 9 (a)	19.42857	13.94872	5	20	2.71089	6.11E-06	REJECT
Figure 9 (b)	23.2	51.55556	5	20	2.71089	9.60E-11	REJECT
Figure 9 (c)	23.17143	50.6875	5	20	2.71089	1.12E-10	REJECT
Figure 10 (a)	179.6657	43.58403	5	995	2.223098	1.05E-40	REJECT
Figure 10 (b)	182.4293	44.40402	5	995	2.223098	2.00E-41	REJECT
Figure 10 (c)	143.9407	33.46054	5	995	2.223098	1.23E-31	REJECT

- The critical value (from the table of critical values for the F distribution for use with ANOVA) with 0.05 significance level and (6, 24) degrees of freedom, is 2.508, so the null hypothesis is rejected at a high level of significance since $F_F = 44.27 >> 2.508$.
- The p-value computed by using $F(6, 24)$ distribution is 8.12658E-12, so the null hypothesis is rejected at a high level of significance since 8.12658E-12 << 0.05

From the obtained results, we can say that the considered voting rules have a very different behavior.

Table 2 summarizes the results of Friedman test and Iman–Davenport extension for different figures considered in this paper. Results suggest that the null hypothesis is always rejected for all settings which means that the *k*-truncated voting rules behave differently.

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